X-730-71-326
PREPRINT

NASA TM X- 65681

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AUGUST 1971

GSFC

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DUAL-SPIN SPACECRAFTS

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ABSTRACT

This work examines the spin-axis attitude stability of dissipative dual-spin spacecrafts in force-free environment. A mathematical model of a representative dual-spin system with energy dissipation on both bodies is proposed and the dynamical equations of motion are derived. The linear variational equations describing the behavior of the perturbed system in a neighborhood of an assumed equilibrium solution are shown to have periodic coefficients. A suitable similarity transformation is introduced which reduces (in the sense of Liapunov) the time-varying system to an autonomous one. This allows all the powerful techniques of time-invariant linear system theory to be applied to the dual-spin system. The theory is applied on a recently developed mathematical model of the SAS-A spacecraft and it is demonstrated that this spacecraft may be analyzed more precisely by utilizing this transformation than what has so far been achieved.

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LIST OF SYMBOLS

0123: reference frame aligned with the principal axes of inertia

A(t): system matrix of an arbitrary time-varying linear system

a(a'): moment arms for dampers on main body (rotor)

B: transformation matrix

c(c'): viscous friction constant for dampers on main body (rotor)

$\hat{c}(\hat{c}')$: c(c') per unit mass

d(d'): axis 3 location of centers of mass of unperturbed dampers on main body (rotor)

h: angular momentum

I: matrix representation of the inertia tensor of the system,

$I = (I_{jk}); j = 1,2,3; k = 1, 2, 3$

I_1, I_2, I_3: principal moments of inertia of the system

$J_3(J_3')$: spin axis moment of inertia of the main body (rotor)

k(k'): spring constant for dampers on the main body (rotor)

$\hat{k}(\hat{k}')$: k(k') per unit mass

$\ell(\ell')$: axis 3 location of center of mass of the main body (rotor)

L_1, L_2 (L_1', L_2'): system parameters defined by $L_1 = ma/I_1$, $L_2 = ma/I_2$, $L_1' = m'a'/I_1$, $L_2' = m'a'/I_2$

M_T: total mass of the system, $M_T = M + M' + 4m + 4m'$

M(M'): mass of the main body (rotor)

m_1, m_2 (m_1', m_2'): masses of dampers on the main body (rotor)

m_3, m_4 (m_3', m_4'): balancing masses on the main body (rotor)

$m(m')$: $m = m_1 = m_2 = m_3 = m_4$ $(m' = m_1' = m_2' = m_3' = m_4')$

n: nominal spin rate of the main body about axis 3

n_1: $n_1 = n + q$

n_2: $n_2 = n + 2q$

$P(t)$: transformation matrix

$Q(s)$: system matrix in the complex frequency domain

s: complex frequency, Laplace transform variable

X: varying part of transverse axes moments of inertia during perturbed motion

y: axis 3 motion of the main body and the rotor during perturbed motion (with respect to the center of mass of the system)

y_1, y_2: transformed variables related to the displacements of the rotor dampers

z_1, z_2 (z_1', z_2'): displacements, with respect to null positions, of dampers on the main body (rotor)

λ_1, λ_2: frequencies

$\omega_1, \omega_2, \omega_3$: system angular rates

Ω: skew symmetric matrix involving angular rates

$\rho(\rho')$: mass ratio, $\rho = m/M_T$ $(\rho' = m'/M_T)$

θ: angular orientation of the rotor, about axis 3, with respect to the main body

INTRODUCTION

It is well known that an ideal rigid body rotating in force-free environment is stable in spin about the axis of either the greatest or the least principal moment of inertia. Where spacecraft applications permit, spin stabilization is an extremely attractive and widely used technique for passive attitude stabilization. Following the launch and subsequent unstable behavior demonstrated by Explorer I in 1958 came the realization that internal energy dissipation may cause the spinning motion about the axis of the least principal moment of inertia to become unstable. To get around this problem spinning spacecrafts are often designed to spin about their axes of the greatest principal moments of inertia. Where this cannot be done for some reason, the so called "dual-spin" configuration may be employed. The basic idea is to attach a relatively large spinning flywheel rigidly to the main spacecraft which may spin, if at all, at a much slower rate. Since the total angular momentum remains inertially fixed in the assumed force-free environment, it is expected that the system spin axis would remain attitude stabilized if the main spacecraft momentum is small.

For the lossless system, it is a simple matter to extend the classical analysis for the Liapunov stability of a single rigid spinning body to the dual-spin case. The main question of interest is of course the stability of the system with the energy dissipation taken into account. An extensive literature exists reporting the results of various design studies of specific dual-spin systems and damper configurations. Not much analytical work has however been done

to explain satisfactorily the role of energy dissipation in causing instability of steady spinning motion. Likins[1] was one of the first to derive a quantitative trade-off relationship between the stabilizing and destabilizing effects of energy dissipations in the two component parts of a dual-spin system. He applied a heuristic energy sink approximation to relate momentum variation with energy dissipation: a very valuable result from a spacecraft designer's point of view. However, he emphatically disclaims rigor and Mingori[2] states that detailed numerical analysis shows that the results obtained from energy sink approximations can in certain cases be obscure or even misleading.

Likins and Mingori[3] discuss some of the considerations involved in a proper choice of variables. Since the inertial attitude stability of the spin axis is of interest, the natural choice is a set of attitude coordinates (e.g., Euler angles) in an inertial frame attached to the center of mass of the system. For the general case of a mechanical system with several connected parts capable of storing angular momenta in various orientations the equations must indeed be formulated in terms of attitude coordinates. Pringle[4] has done this to develop a general stability theory of mechanical systems but it is not quite clear if his theory can be applied to a dual-spin system with energy dissipation on both bodies.

The equations of motion are more conveniently written in terms of the angular rates resolved along a body fixed reference frame. It can be easily seen that for the specific case of the dual-spin configuration, some form of Euler's equations in terms of angular rates suitably augmented by energy dissipation equations is an

adequate mathematical model. In this case, the equilibrium motion is a spin about one of the body axes which is the common axis of spin of all the connected parts of the system. In stable spin equilibrium the total system angular momentum is fixed in the body and inertial attitude stability of the spin axis may be inferred by an appeal to the fact that this momentum vector is inertially fixed in the assumed force-free environment.

Accurate modeling is extremely difficult to achieve for the slightly flexible dissipative spacecraft structures and some of the various damping mechanisms that have been proposed or used in dual-spin systems. Precise stability results can however be obtained only for mathematically well defined problems. We therefore look for a model which is of some practical interest in itself and is also general enough to serve as an approximate model for a large class of dual-spin systems. Mingori[2] has considered such a model and we take a somewhat more general version of his system as our basic dissipative dual-spin configuration. In this model, the energy dissipation in any component part, whether inherent or introduced through a damping mechanism, is modeled as produced by one or more ball-in-tube dampers. This damper is a practical realization of an ideal linear damped harmonic oscillator and consists of a spring-restrained point mass in a short straight viscous fluid filled tube. The generalization from Mingori's model lies in the appreciation of the fact that the high speed rotor is always made symmetric and the rotor energy dissipation must therefore be symmetric in the two transverse axes.

The equations of motion of the basic dissipative dual-spin system are highly nonlinear. Rigorous approaches, such as an attempt to construct a Liapunov function, do not appear to be very promising. The natural attack on the problem is to apply the Principle of Stability in the First Approximation[5] which succeeds easily if the linearized variational system of equations of perturbed motion is time-invariant and does not have purely imaginary characteristic roots. It turns out however that the linearized equations for this system have periodic coefficients. As Hahn[5] points out, the known stability results for such systems are not exactly stability criteria because in order to apply them several things must already be known about the solutions.

An exact method of analysis may be developed by use of Floquet's theorem and the Liapunov Reducibility theorem.[5] Each linear differential equation with periodic coefficients can be reduced (in the sense of Liapunov) to a kinematically similar autonomous differential equation having the same stability behavior by use of a nonsingular matrix with periodic coefficients. Stability and performance analysis of the reduced system is a fairly routine exercise in linear system theory. The main difficulty in applying this approach is of course finding the appropriate similarity transformation. For the proposed dissipative dual-spin model physical insight suggests a very simple quasi-holonomic transformation which serves the purpose.

MATHEMATICAL MODEL OF THE BASIC DISSIPATIVE DUAL-SPIN CONFIGURATION

Consider the model in Fig. 1 which shows an asymmetrical rigid body of mass M, which will be called the main body, carrying a uniaxial rotor of mass M' spinning about a principal axis. Assume that the energy dissipation on the rotor can be modelled by two ball-in-tube dampers of mass m_1' and m_2', spring constant c' and viscous friction coefficient k', each with a single degree of freedom in a direction parallel to the wheel axis. Let there be two other mass points m_3' and m_4' for static and dynamic balance of the system around the spin axis. To counteract the destabilizing effect of rotor dissipation, let there be two similar dampers with masses m_1 and m_2, spring constant c and viscous friction coefficient k, as well as two balancing masses m_3 and m_4 on the main body. The configuration is more precisely described by the mathematical formulation below.

Let 0 be the center of mass of the system, which is fixed in inertial space, and consider a rectangular frame 0123 along the principal axes of inertia of the system at rest. Let 3 be the spin axis. Let θ be the angular orientation of the rotor relative to the main body and assume that an ideal servo maintains $\dot{\theta}$ constant at a value q. The table below specifies the centers of masses of the various bodies both when the system is at rest and when it is undergoing perturbed motion.

Table 1. Configuration of the Basic Dual-Spin System

Mass	Rest Position			Perturbed Position		
M	0	0	ℓ	0	0	$\ell + y$
M'	0	0	ℓ'	0	0	$\ell' + y$
m_1	a	0	d	a	0	$d + y + z_1$
m_2	0	a	d	0	a	$d + y + z_2$
m_3	$-a$	0	d	$-a$	0	$d + y$
m_4	0	$-a$	d	0	$-a$	$d + y$
m_1'	a'	0	d'	$a' \cos\theta$	$a' \sin\theta$	$d + y + z_1'$
m_2'	0	a'	d'	$-a' \sin\theta$	$a' \cos\theta$	$d' + y + z_2'$
m_3'	$-a'$	0	d'	$-a' \cos\theta$	$-a' \sin\theta$	$d' + y$
m_4'	0	$-a'$	d'	$a' \sin\theta$	$-a' \cos\theta$	$d' + y$

For simplicity, we have assumed that $m_1 = m_2 = m_3 = m_4 = m$ and $m_1' = m_2' = m_3' = m_4' = m'$. The damper mass displacements from their respective null positions are z_1, z_2, z_1' and z_2'. Equating moments about the origin

$$y = -\rho (z_1 + z_2) - \rho' (z_1' + z_2') \tag{1}$$

where $\rho = m/M_T$, $\rho' = m'/M_T$ and $M_T = M + M' + 4m + 4m'$ is the total mass of the system.

Let I_1, I_2 and I_3 be the principal moments of inertia of the system at rest. Write $I_3 = J_3 + J_3'$ where J_3 and J_3' are the moments of inertia of the main body and the rotor, each including the corresponding dampers and balancing masses at rest. During perturbed motion, the components of the inertia tensor are

$$I_{11} = I_1 + X$$

$$I_{22} = I_2 + X$$

$$I_{33} = I_3 = J_3 + J_3'$$

$$I_{12} = I_{21} = 0$$

$$I_{23} = I_{32} = -ma\, z_2 - m'a' \, (z_1' \sin\theta + z_2' \cos\theta)$$

$$I_{13} = I_{31} = -ma\, z_1 - m'a' \, (z_1' \cos\theta - z_2' \sin\theta)$$

where

$$X = M_T y^2 + m \left\{z_1^2 + z_2^2 + 2(z_1 + z_2)\,(d + y)\right\} + m' \left\{z_1'^2 + z_2'^2 + 2(z_1' + z_2')\,(d' + y)\right\} \tag{2}$$

The angular momentum of the system with the rotor and the dampers frozen in is given by $I\omega$ where I is the matrix representation of the inertia tensor defined above and ω is the angular velocity vector, $\omega = \mathrm{col}(\omega_1, \omega_2, \omega_3)$. The rotor contributes an additional momentum of $J_3' q$ along the third axis. The calculation of the inertial I already takes into account the effects of much of the linear momenta of the damper masses, the additional terms to be considered are due to the time variations of the damper displacements and wheel rotation. From basic mechanics, a unit mass point at position

$x = \mathrm{col}\,(x_1, x_2, x_3)$ on a body with angular velocity $\omega = \mathrm{col}\,(\omega_1, \omega_2, \omega_3)$ contributes to the total angular momentum

$$\Delta h = \begin{bmatrix} x_2\dot{x}_3 - x_3\dot{x}_2 + \omega_1\,(x_2^2 + x_3^2) - \omega_2 x_1 x_2 - \omega_3 x_1 x_3 \\ x_3\dot{x}_1 - x_1\dot{x}_3 - \omega_1 x_2 x_1 + \omega_2\,(x_3^2 + x_1^2) - \omega_3 x_2 x_3 \\ x_1\dot{x}_2 - x_2\dot{x}_1 - \omega_1 x_3 x_1 - \omega_2 x_3 x_2 + \omega_3\,(x_1^2 + x_2^2) \end{bmatrix} \tag{3}$$

We sum the above expression over the dampers neglecting the terms already taken into account and add to $I\omega$ to get the total angular momentum of the system as

$$h = \begin{bmatrix} I_1\omega_1 + X\omega_1 + ma\,(\dot{z}_2 - \omega_3 z_1) \\ \qquad + m'a'\left\{\cos\theta\,(\dot{z}_2' - (\omega_3 + q)\,z_1') + \sin\theta\,(\dot{z}_1' + (\omega_3 + q)\,z_2')\right\} \\ I_2\omega_2 + X\omega_2 - ma\,(\dot{z}_1 + \omega_3 z_2) \\ \qquad - m'a'\left\{-\sin\theta\,(\dot{z}_2' - (\omega_3 + q)\,z_1') + \cos\theta\,(\dot{z}_1' + (\omega_3 + q)z_2')\right\} \\ I_3\omega_3 + J_3' q - ma\,(\omega_1 z_1 + \omega_2 z_2) \\ \qquad - m'a'\left\{\cos\theta\,(\omega_1 z_1' + \omega_2 z_2') - \sin\theta\,(\omega_1 z_2' - \omega_2 z_1')\right\} \end{bmatrix} \tag{4}$$

Inertial differentiation of the angular momentum yields the rotational equations of motion as

$$\begin{aligned} I_1\dot{\omega}_1 &- \left\{(I_2 - I_3)\,\omega_3 - J_3' q\right\}\omega_2 + X\,(\dot{\omega}_1 - \omega_2\omega_3) + \dot{X}\omega_1 \\ &+ ma\left\{\ddot{z}_2 - (\dot{\omega}_3 + \omega_1\omega_2)\,z_1 - (\omega_2^2 - \omega_3^2)\,z_2\right\} \\ &+ m'a'\Big\{\cos\theta\left[\ddot{z}_2' - (\dot{\omega}_3 + \omega_1\omega_2)\,z_1' - \left\{\omega_2^2 - (\omega_3 + q)^2\right\} z_2'\right] \\ &\quad + \sin\theta\left[\ddot{z}_1' + (\dot{\omega}_3 + \omega_1\omega_2)\,z_2' - \left\{\omega_2^2 - (\omega_3 + q)^2\right\} z_1'\right]\Big\} = 0 \end{aligned} \tag{5}$$

$$
\begin{aligned}
I_2 \dot{\omega}_2 &- \left\{(I_3 - I_1)\,\omega_3 + J_3'\,q\right\}\omega_1 + X(\dot{\omega}_2 + \omega_3\omega_1) + \dot{X}\omega_2 \\
&- ma\,\left\{\ddot{z}_1 + (\dot{\omega}_3 - \omega_1\omega_2)\,z_2 + (\omega_3^2 - \omega_1^2)\,z_1\right\} \\
&- m'a'\left\{\cos\theta\left[\ddot{z}_1' + (\dot{\omega}_3 - \omega_1\omega_2)\,z_2' + \left\{(\omega_3 + q)^2 - \omega_1^2\right\}z_1'\right]\right. \\
&\qquad \left. - \sin\theta\left[\ddot{z}_2' - (\dot{\omega}_3 - \omega_1\omega_2)\,z_1' + \left\{(\omega_3 + q)^2 - \omega_1^2\right\}z_2'\right]\right\} = 0 \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
I_3\dot{\omega}_3 &- (I_1 - I_2)\,\omega_1\omega_2 \\
&- ma\left\{2\omega_1\dot{z}_1 + 2\omega_2\dot{z}_2 + (\dot{\omega}_1 - \omega_2\omega_3)\,z_1 + (\dot{\omega}_2 + \omega_3\omega_1)\,z_2\right\} \\
&- m'a'\left\{\cos\theta\left[2\omega_1\dot{z}_1' + 2\omega_2\dot{z}_2' + (\dot{\omega}_1 - \omega_2\omega_3)\,z_1' + (\dot{\omega}_2 + \omega_3\omega_1)\,z_2'\right]\right. \\
&\qquad \left. - \sin\theta\left[2\omega_1\dot{z}_2' - 2\omega_2\dot{z}_1' + (\dot{\omega}_1 - \omega_2\omega_3)\,z_2' - (\dot{\omega}_2 + \omega_3\omega_1)\,z_1'\right]\right\} = 0 \qquad (7)
\end{aligned}
$$

These equations have to be augmented by four others for the four dampers. From the basic rule for inertial differentiation,[6] a mass point with position $x = \mathrm{col}\,(x_1, x_2, x_3)$ has a linear acceleration in inertial space given by

$$\ddot{x} + 2\Omega\dot{x} + (\dot{\Omega} + \Omega^2)\,x \qquad (8)$$

where

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad (9)$$

The equation of motion for any damper is obtained by summing the third component of the vector expression (8) with the corresponding viscous and elastic forces and equating to zero. Thus

$$m\left\{\ddot{y} + \ddot{z}_1 - a\,(\dot{\omega}_2 - \omega_3\omega_1) - (\omega_1^2 + \omega_2^2)\,(d + y + z_1)\right\} + c\dot{z}_1 + kz_1 = 0 \qquad (10)$$

$$m\left\{\ddot{y} + \ddot{z}_2 + a(\dot{\omega}_1 + \omega_2\omega_3) - (\omega_1^2 + \omega_2^2)\,(d + y + z_2)\right\} + c\dot{z}_2 + kz_2 = 0 \tag{11}$$

$$\begin{aligned} m'\Big\{ \ddot{y} + \ddot{z}_1' &+ a' \cos\theta \left[2q\omega_1 - (\dot{\omega}_2 - \omega_3\omega_1)\right] \\ &+ a' \sin\theta \left[2q\omega_2 + (\dot{\omega}_1 + \omega_2\omega_3)\right] \\ &- (\omega_1^2 + \omega_2^2)\,(d' + y + z_1')\Big\} + c'\,\dot{z}_1' + k'\,z_1' = 0 \end{aligned} \tag{12}$$

$$\begin{aligned} m'\Big\{ \ddot{y} + \ddot{z}_2' &+ a' \cos\theta \left[2q\omega_2 + (\dot{\omega}_1 + \omega_2\omega_3)\right] \\ &- a' \sin\theta \left[2q\omega_1 - (\dot{\omega}_2 - \omega_3\omega_1)\right] \\ &- (\omega_1^2 + \omega_2^2)\,(d' + y + z_2')\Big\} + c'\,\dot{z}_2' + k'\,z_2' = 0 \end{aligned} \tag{13}$$

In trying to apply the Principle of Stability in the First Approximation, we linearize the system equations of motion in a neighborhood of the desired equilibrium point defined by

$$\begin{aligned} \omega_3 &= n, \text{ constant} \\ \omega_1 = \omega_2 = z_1 = z_2 &= z_1' = z_2' = 0 \end{aligned} \tag{14}$$

We substitute $\omega_3 = n + \nu_3$ and from the linearized version of (28), see that $\dot{\nu}_3 = 0$. Without loss of generality, we put $\nu_3 = 0$. The other equations, when linearized, yield

$$\begin{aligned} I_1\,\dot{\omega}_1 - \left\{(I_2 - I_3)\,n - J_3'\,q\right\}\omega_2 &+ ma\,(\ddot{z}_2 + n^2\,z_2^2) \\ &+ m'a' \left\{\cos\theta\,(\ddot{z}_2' + (n+q)^2 z_2') + \sin\theta(\ddot{z}_1' + (n+q)^2 z_1')\right\} = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} I_2\dot{\omega}_2 - \left\{(I_3 - I_1)\,n + J_3'\,q\right\}\omega_1 &- ma\,(\ddot{z}_1 + n^2\,z_1) \\ &- m'a' \left\{\cos\theta(\ddot{z}_1' + (n+q)^2 z_1') - \sin\theta(\ddot{z}_2' + (n+q)^2\,z_2')\right\} = 0 \end{aligned} \tag{16}$$

$$m\left\{\ddot{y} + \ddot{z}_1 - a(\dot{\omega}_2 - n\omega_1)\right\} + c\dot{z}_1 + kz_1 = 0 \tag{17}$$

$$m\left\{\ddot{y} + \ddot{z}_2 + a(\dot{\omega}_1 + n\omega_2)\right\} + c\dot{z}_2 + kz_2 = 0 \tag{18}$$

$$\begin{aligned} m'\Big\{\ddot{y} + \ddot{z}_1' - a'\cos\theta\,(\dot{\omega}_2 - (n+2q)\,\omega_1) \\ + a'\sin\theta\,(\dot{\omega}_1 + (n+2q)\,\omega_2)\Big\} + c'\dot{z}_1' + k'z_1' = 0 \end{aligned} \tag{19}$$

$$\begin{aligned} m'\Big\{\ddot{y} + \ddot{z}_2' + a'\cos\theta\,(\dot{\omega}_1 + (n+2q)\,\omega_2) \\ + a'\sin\theta\,(\dot{\omega}_2 - (n+2q)\,\omega_1)\Big\} + c'\dot{z}_2' + k'z_2' = 0 \end{aligned} \tag{20}$$

The stability of the origin of this set of equations has to be examined.

A SIMILARITY TRANSFORMATION FOR LIAPUNOV REDUCIBILITY

Assume that the system of equations (15)-(20) is put in a standard vector-matrix first order differential equation form

$$\dot{x} = A(t)\,x \tag{21}$$

where x is an appropriately defined state vector in R^{10} and $A(t)$ is the 10×10 system matrix which has periodic coefficients. The Liapunov Reducibility Theorem[5] asserts that there exists a bounded nonsingular transformation $P(t)$ with periodic coefficients with a bounded inverse $P^{-1}(t)$ such that the matrix B defined by the relation

$$B = P^{-1}AP - P^{-1}\dot{P} \tag{22}$$

is a constant matrix. We may then define

$$x = P(t)\,\hat{x} \tag{23}$$

and obtain

$$\dot{\hat{x}} = B\hat{x} \tag{24}$$

The autonomous system defined by (24) is kinematically similar to the original system (21) and has similar stability behavior.[5] We may therefore confine our attention to the examination of the stability of (24). This is easy to do because the Routh-Hurwitz criterion provides a well defined algorithm.

The handling of the tenth order system matrix defined in (21) can be clumsy and we would rather use the scalar equations (15)-(20) directly. To simplify the equations we introduce the approximation that y is identically zero. It can be seen from the definition of y in (1) that this is equivalent to assuming that the mass of any particular damper is negligible compared to the total mass of the system. This assumption implies that the location of the center of mass of the system is fixed in the body and is unaffected by the vibrations of the damper masses. We also define

$$\lambda_1 = \frac{I_3 n + J_3' q - I_1 n}{I_2}$$

$$\lambda_2 = \frac{I_3 n + J_3' q - I_2 n}{I_1}$$

$$\frac{ma}{I_1} = L_1; \quad \frac{m'a'}{I_1} = L_1'; \quad \frac{ma}{I_2} = L_2; \quad \frac{m'a'}{I_2} = L_2'$$

$$c/m = \hat{c}; \quad c'/m' = \hat{c}'; \quad k/m = \hat{k}; \quad k'/m' = \hat{k}',$$

$$n + q = n_1; \quad n + 2q = n_2 \tag{25}$$

The equations (15)-(20), may now be written

$$\dot{\omega}_1 + \lambda_2\omega_2 + L_1(\ddot{z}_2 + n^2 z_2) + L_1' \left\{\cos\theta(\ddot{z}_2' + n_1^2 z_2') + \sin\theta(\ddot{z}_1' + n_1^2 z_1')\right\} = 0 \tag{26}$$

$$\dot{\omega}_2 - \lambda_1\omega_1 - L_2(\ddot{z}_1 + n^2 z_1) - L_2' \left\{\cos\theta(\ddot{z}_1' + n_1^2 z_1') - \sin\theta(\ddot{z}_2' + n_1^2 z_2')\right\} = 0 \tag{27}$$

$$\ddot{z}_1 - a(\dot{\omega}_2 - n\omega_1) + \hat{c}\dot{z}_1 + \hat{k}z_1 = 0 \tag{28}$$

$$\ddot{z}_2 + a(\dot{\omega}_1 + n\omega_2) + \hat{c}\dot{z}_2 + \hat{k}z_2 = 0 \tag{29}$$

$$\ddot{z}_1' - a'\cos\theta(\dot{\omega}_2 - n_2\omega_1) + a'\sin\theta(\dot{\omega}_1 + n_2\omega_2) + \hat{c}'\dot{z}_1' + \hat{k}'z_1' = 0 \tag{30}$$

$$\ddot{z}_2' - a'\cos\theta(\dot{\omega}_1 + n_2\omega_2) + a'\sin\theta(\dot{\omega}_2 - n_2\omega_1) + \hat{c}'\dot{z}_2' + \hat{k}'z_2' = 0 \tag{31}$$

Introducing the nonholonomic transformation

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{32}$$

we have

$$\dot{\omega}_1 + \lambda_2\omega_2 + L_1(\ddot{z}_2 + n^2 z_2) + L_1'(\ddot{y}_2 - 2q\dot{y}_1 + n n_2 y_2) = 0 \tag{33}$$

$$\dot{\omega}_2 - \lambda_1\omega_1 - L_2(\ddot{z}_1 - n^2 z_1) - L_2'(\ddot{y}_1 + 2q\dot{y}_2 + n n_2 y_1) = 0 \tag{34}$$

$$\ddot{z}_1 - a(\dot{\omega}_2 - n\omega_1) + \hat{c}\,\dot{z}_1 + \hat{k}\,z_1 = 0 \qquad (35)$$

$$\ddot{z}_2 + a(\dot{\omega}_1 + n\omega_2) + \hat{c}\,\dot{z}_2 + \hat{k}\,z_2 = 0 \qquad (36)$$

$$\ddot{y}_1 + 2q\,\dot{y}_2 - q^2 y_1 - a'(\dot{\omega}_2 - n_2\,\omega_1) + \hat{c}'(\dot{y}_1 + qy_2) + \hat{k}'\,y_1 = 0 \qquad (37)$$

$$\ddot{y}_2 - 2q\,\dot{y}_1 - q^2 y_2 + a'(\dot{\omega}_1 + n_2\,\omega_2) + \hat{c}'(\dot{y}_2 - qy_1) + \hat{k}'\,y_2 = 0 \qquad (38)$$

(37) is obtained by multiplying (30) by $\cos\theta$ and (31) by $-\sin\theta$ and adding; (38) is similarly obtained by multiplying (30) by $\sin\theta$ and (31) by $\cos\theta$ and adding.

The last six equations form a set of simultaneous linear autonomous differential equations. Taking Laplace transform, the characteristic matrix of the system in the complex frequency s-domain is

$$Q(s) = \begin{bmatrix} s & \lambda_2 & 0 & L_1(s^2+n^2) & -2qL_1' s & L_1'(s^2+n\,n_2) \\ -\lambda_1 & s & -L_2(s^2+n^2) & 0 & -L_2'(s^2+n\,n_2) & -2qL_2' s \\ an & -as & s^2+\hat{c}s+\hat{k} & 0 & 0 & 0 \\ as & an & 0 & s^2+\hat{c}s+\hat{k} & 0 & 0 \\ a'n_2 & -a's & 0 & 0 & s^2-q^2+\hat{c}'s+\hat{k}' & q(2s+\hat{c}') \\ a's & a'n_2 & 0 & 0 & -q(2s+\hat{c}') & s^2-q^2+\hat{c}'s+\hat{k}' \end{bmatrix} \qquad (39)$$

The characteristic equation of the system is obtained by setting the determinant of the above matrix to zero:

$$|Q(s)| = 0 \qquad (40)$$

The problem of the determination of system stability is, in principle, solved and requires only a routine computation.

The transformation may clearly be put in the more elegant form suggested in (21). However, the above presentation is expected to bring out the basic idea more distinctly. The matrix in (32) simply defines a rotation operation and is extremely well known and widely used. In the present context, this transformation effectively replaces the two rotor dampers by two equivalent ones which are, as it were, fixed in the body frame. The mutual coupling terms therefore have constant coefficients.

AN EXAMPLE: THE SAS-A SPACECRAFT

SAS-A is a dual-spin spacecraft launched at the end of 1970. It has been subjected to exhaustive study, a recent work being that of Bainum.[7] It has a pendulus nutation damper on the main body and a slightly flexible momentum wheel. Bainum takes axis 2 (Y axis) as the common spin axis and derives the nonlinear equations of motion. When linearized in a neighborhood of the assumed equilibrium motion, the following equations are obtained:

$$\overline{B}\dot{\nu}_2 + m r_1 (r_1 + r_0)\ddot{\phi} = 0 \tag{41}$$

$$\begin{aligned}&\overline{A}\dot{\omega}_1 + \Omega\omega_3 (\overline{C} - \overline{B}) - \omega_3 I_{R_2} s - 2 m r_1 \frac{\ell M}{\overline{M}} \Omega\dot{\phi} \\ &+ I_{R_1} \{ [\ddot{\alpha}_y + (\Omega + s)^2 \alpha_y] \cos\theta + [\ddot{\alpha}_z + (\Omega + s)^2 \alpha_z] \sin\theta \} = 0\end{aligned} \tag{42}$$

$$\bar{C}\dot{\omega}_3 + \Omega\omega_1 (\bar{B} - \bar{A}) + \omega_1 I_{R_2} s - m r_1 \frac{\ell M}{\bar{M}} \ddot{\phi} + m r_1 \frac{\ell M}{\bar{M}} \Omega^2 \phi$$

$$+ I_{R_1} \{- [\ddot{\alpha}_y + (\Omega + s)^2 \alpha_y] \sin\theta + [\ddot{\alpha}_z + (\Omega + s)^2 \alpha_z] \cos\theta\} = 0 \quad (43)$$

$$mr_1^2 \left(1 - \frac{m}{\bar{M}}\right) \ddot{\phi} - mr_1 \frac{\ell M}{\bar{M}} \dot{\omega}_3 + mr_1 (r_0 + r_1) \dot{\nu}_2 + mr_1 \left(r_0 + \frac{mr_1}{\bar{M}}\right) \Omega^2 \phi$$

$$+ mr_1 \frac{\ell M}{\bar{M}} \Omega\omega_1 + k\dot{\phi} + K\phi = 0 \quad (44)$$

$$I_{R_1} \ddot{\alpha}_z + k_R \dot{\alpha}_z + [K_R + I_{R_1} (\Omega + s)^2] \alpha_z$$

$$+ I_{R_1} \{[\dot{\omega}_3 + (\Omega + 2s) \omega_1] \cos\theta + [\dot{\omega}_1 - (\Omega + 2s) \omega_3] \sin\theta\} = 0 \quad (45)$$

$$I_{R_1} \ddot{\alpha}_y + k_R \dot{\alpha}_y + [K_R + I_{R_1} (\Omega + s)^2] \alpha_y$$

$$+ I_{R_1} \{- [\dot{\omega}_3 + (\Omega + 2s)\omega_1] \sin\theta + [\dot{\omega}_1 - (\Omega + 2s)\omega_3] \cos\theta\} = 0 \quad (46)$$

where Bainum's [7] notation has been used with the addition of

$$\omega_2 = \Omega + \nu_2, \ \Omega \text{ constant}$$

$$\theta = st$$

$$\phi = \phi_1$$

$$\alpha_y = -\alpha_x \quad (47)$$

Eq. (41) may be eliminated by solving for $\dot{\nu}_2$ and substituting in (44).

Bainum introduces the following notations and approximations

$$\bar{A} = \bar{C} = A$$

$$\bar{B} = B$$

$$\Gamma = \ell M/\bar{M}$$

$$\lambda = \{\Omega(B - A) + I_{R_2}\, s\}/A$$

$$m/\bar{M} \ll 1 \tag{48}$$

We introduce a transformation

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha_z \\ \alpha_y \end{bmatrix} \tag{49}$$

The reduced set of kinematically similar autonomous equations are

$$\left\{1 - \frac{m(r_0 + r_1)^2}{B}\right\}\ddot{\phi} + \frac{k}{m r_1^2}\dot{\phi} + \left\{\frac{K}{m r_1^2} + \frac{r_0\,\Omega^2}{r_1}\right\}\phi$$

$$-\frac{\Gamma}{r_1}\dot{\omega}_3 + \frac{\Gamma}{r_1}\Omega\omega_1 = 0 \tag{50}$$

$$\frac{2 m r_1 \Gamma}{I_{R_1}}\Omega\dot{\phi} + \frac{\lambda A}{I_{R_1}}\omega_3 - \frac{A}{I_{R_1}}\dot{\omega}_1 + 2s\dot{\beta}_1 - \{\ddot{\beta}_2 + \Omega(\Omega + 2s)\beta_2\} = 0 \tag{51}$$

$$-\frac{m r_1 \Gamma}{I_{R_1}}\{\ddot{\phi} - \Omega^2\phi\} + \frac{A}{I_{R_1}}\dot{\omega}_3 + \frac{\lambda A}{I_{R_1}}\omega_1$$

$$+\{\ddot{\beta}_1 + \Omega(\Omega + 2s)\beta_1\} + 2s\dot{\beta}_2 = 0 \tag{52}$$

$$\dot{\omega}_3 + (\Omega + 2s)\,\omega_1 + \ddot{\beta}_1 + \frac{k_R}{I_{R_1}}\dot{\beta}_1 + \left[\frac{K_R}{I_{R_1}} + \Omega(\Omega + 2s)\right]\beta_1 + 2s\left[\dot{\beta}_2 + \frac{k_R}{2\,I_{R_1}}\beta_2\right] = 0 \quad (53)$$

$$-(\Omega + 2s)\,\omega_3 + \dot{\omega}_1 - 2s\left[\dot{\beta}_1 + \frac{k_R}{2\,I_{R_1}}\beta_1\right]$$

$$+\ddot{\beta}_2 + \frac{k_R}{I_{R_1}}\dot{\beta}_2 + \left[\frac{K_R}{I_{R_1}} + \Omega(\Omega + 2s)\right]\beta_2 = 0 \quad (54)$$

The study of this set of equations is a standard exercise in linear systems theory. In particular, the stability of t, origin for the nominal parameter values may be easily studied by Routh's method. In the above, Ω and s represent respectively the main spacecraft spin rate and the rotor relative spin rate. We replace these by the symbols n and q respectively to be consistent with the notation used before. These are the only two parameters in the above equations that are adjustable: the nominal values being 1/12 RPM and 2000 RPM respectively. Using the values given by Bainum[7] for the other parameters, and denoting the Laplace Transform complex frequency variable by s, we have the characteristic matrix Q(s) of the system as

$$
Q(s) = \begin{bmatrix}
\begin{matrix} s^2 + 0.0097s \\ + 0.0084 + 0.1n^2 \end{matrix} & -2.4s & 2.4n & 0 & 0 \\[2ex]
6.04\,ns & 267n + 2s & -4688s & 2qs & \begin{matrix} -s^2 \\ -n(n + 2q) \end{matrix} \\[2ex]
-3.02(s^2 - n^2) & 4688s & \begin{matrix} 267n \\ + 2s \end{matrix} & \begin{matrix} s^2 \\ + n(n + 2q) \end{matrix} & 2qs \\[2ex]
0 & s & n + 2q & \begin{matrix} s^2 + 1.18s \\ + 12433 \\ + n(n + 2q) \end{matrix} & \begin{matrix} 2qs \\ + 1.18 \end{matrix} \\[2ex]
0 & -(n + 2q) & s & \begin{matrix} -2qs \\ -1.18 \end{matrix} & \begin{matrix} s^2 + 1.18s \\ + 12433 \\ + n(n + 2q) \end{matrix}
\end{bmatrix} \tag{55}
$$

The characteristic polynomial $|Q(s)|$ may be evaluated for various assumed values of n and q using a digital computer. Application of Routh's criterion shows that the system is stable for a reasonably wide range of spacecraft and rotor spin rates around the nominal values. This result is hardly surprising: flight data has shown the system to be stable and a number of preflight simulation and experimental studies and approximate analyses had drawn the same conclusion. This, however, is the first analytic proof of stability under steady state nominal and off-nominal operating conditions and depends only on the validity of the mathematical model.

CONCLUSIONS

It has been shown that as required by the Floquet Theory and the Liapunov Reducibility Theorem,[5] a similarity transformation does exist which reduces the linear time-varying system of variational equations of a dual-spin system to a linear time-invariant system and thus makes a complete and rigorous analysis possible. The analysis has been applied to a representative mathematical model of a dissipative dual-spin system and also to a recent model[7] of the SAS-A spacecraft. The method of analysis makes use of the fact that most dual-spin spacecrafts have symmetrical rotor dissipations in the two transverse axes.

It is evident that more detailed analytical studies can be made by use of the proposed approach. Conclusive proofs for steady state spin stability may be obtained. Moreover, since the transformation is relatively easy to apply, the steady state spin stability for various assumed spin rates over a wide range of values may be studied. If the property of quasi-stationarity is assumed (frozen-time approximation), the above results may be used to analyze stability during slow spin-up, a problem not considered in any detail in present designs.

REFERENCES

1. Likins, P. W., "Attitude Stability Criteria for Dual Spin Spacecraft," Journal of Spacecraft and Rockets, Vol. 4, No. 12, Dec. 1967, pp. 1638-1643.

2. Mingori, D. L., "Effects of Energy Dissipation on the Attitude Stability of Dual Spin Satellites," AIAA Journal, Vol. 7, No. 1, Jan. 1969, pp. 20-27.

3. Likins, P. W. and D. L. Mingori, "Liapunov Stability Analysis of Freely Spinning Systems," Proceedings of the International Astronautical Federation Congress, Belgrade, 1967, pp. 89-102.

4. Pringle, R., Jr., "Stability of the Force-Free Motions of a Dual-Spin Space-Craft," AIAA Journal, Vol. 7, No. 6, June 1969, pp. 1054-1063.

5. Hahn, W., Stability of Motion, Springer-Verlag, 1967, pp. 120-122, 296-304.

6. Whittaker, E. T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press, 1964, pp. 15-18, 124-126.

7. Bainum, P. M., "Stability of SAS-A Dual Spin Spacecraft with Energy Dissipation on the Momentum Wheel," NASA Goddard Space Flight Center Preprint, X-732-70-344, Sept. 1970.

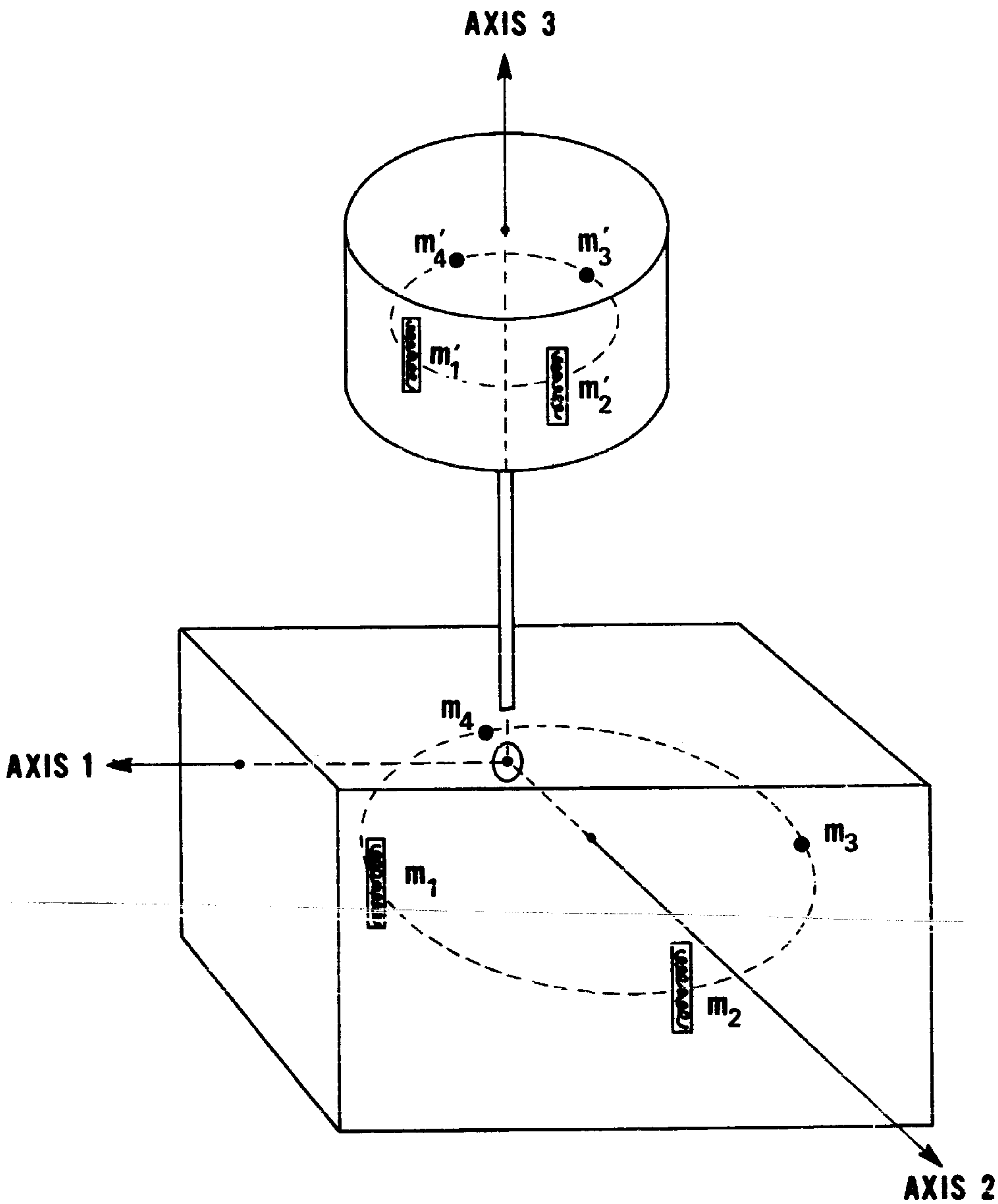

Figure 1. Basic Dual Spin Configuration